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ORIGINAL ARTICLE

On the on-line rent-or-buy problem in probabilistic environments

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Abstract Fujiwara and Iwama [In: The 13th Annual International Symposium on Algorithms and Computation, pp. 476–488 (2002)] first integrated probability distribution into the classical competitive analysis to study the rental problem. They assumed that the future inputs are drawn from an exponential distribution, and obtained the optimal competitive strategy and the competitive ratio by the derivative method. In this paper, we introduce the interest rate and tax rate into the continuous model of Fujiwra and Iwama [In: The 13th Annual International Symposium on Algorithms and Computation, pp. 476–488 (2002)]. Moreover, we use the forward difference method in different probabilistic environments to consider discrete leasing models both with and without the interest rate. We not only give the optimal competitive strategies and their competitive ratios in theory, but also give numerical results. We find that with the introduction of the interest rate and tax rate, the uncertainty involved in the process of decision making will diminish and the optimal purchasing date will be put off.

Keywords On-line algorithm · Rent-or-buy problem · Probabilistic distribution · Competitive analysis

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1 Introduction

In this paper, we consider improving the performance measure of competitive analysis by integrating more information of the input structures into pure competitive analysis. We introduce the interest rate and tax rate into the continuous model of Fujiwra and Iwama [10]. Moreover, we use the forward difference method in different probabilistic environments to consider discrete leasing models both with and without the interest rate. We not only give the optimal competitive strategies and their competitive ratios in theory, but also give some numerical results. It could be found that the introduction of an interest rate and a tax rate would diminish the uncertainty involved in the process of decision making and put off the optimal purchasing date. This is only a small move to obtain a more realistic solution of the problem; however, the analysis is much more complicated because of the introduction of new parameters. There are also other interesting insights not found in the Fujiwra's and Iwama's model with no interest rate and tax rate. Therefore, we generalize the Fujiwra and Iwama model with significant results.

1.1 Foundations

Many economic and financial decisions that are subject to uncertainty are conducted in an ongoing fashion. For example, decisions related to currency exchange, stock transactions or mortgage financing, etc., have to be made on-line. There are two approaches to the problem: the classical approach and the competitive analysis. The classical approach to studying the performance of on-line decision-making processes (algorithms) relies on the use of probabilistic analysis. In the classical approach, decision-makers faced by uncertainty often have two different types of models. The first type of model makes assumptions about the future distribution of relevant variables such as exchange rates or mortgage rates, and aims for an acceptable average outcome. The second type of model focuses on the worst case and then makes a decision. It is well recognized in the literature that these two types of models may give unrealistic on-line algorithms that are far from optimal. Furthermore, for many real-life problems, an adequate stochastic model is extremely difficult or costly to devise.

In competitive analysis (which was first applied to on-line algorithms by Sleator and Tarjian [20]), we contrast the performance of an on-line strategy with that of an optimal off-line strategy. We assume that the optimal off-line strategy has full knowledge of future events. We minimize the worst-case ratio of on-line cost to optimal cost or of optimal profit to on-line profit, for some measure of cost or of profit. Note that if this ratio is bounded for all event sequences, then the on-line strategy is competitive. We define the competitive ratio of this strategy to be the supremum of this ratio for the profit measure and the infimum of this ratio for the cost measure.

Specifically, an on-line algorithm incrementally receives one observation (the input) in each time period. An output is generated through the algorithm without knowledge of the values of future observations. In a competitive analysis, an on-line algorithm A is compared to an optimal off-line algorithm OPT. The optimal off-line algorithm has full information of the entire input sequence and can act optimally. The performance measure has an advantage over the traditional average-case measure. For most nontrivial decision-making activities it is analytically difficult for the traditional average-result measure to come up with an accurate probabilistic model solution [15,16].

Consider a cost problem. For a given input sequence I, denote $C_A(I)$ and $C_{\mathrm{OPT}}(I)$ as the costs incurred by A and OPT in process I, respectively. Algorithm A is called α-competitive if there are constants α and β such that,

$$C_A(I) \leq \alpha \cdot C_{\mathrm{OPT}}(I) + \beta, \tag{1}$$

for all input sequences I. For a competitive algorithm it must perform well on all input sequences. A similar definition can be given for on-line profit maximization problems.

From the above definition, it is clear that the competitive ratio is a worst case performance measure. Therefore, intuitively we can view an on-line problem as a two person (zero-sum) game between an adversary and the on-line player. In general, randomization is required to obtain optimal (expected) competitive performance. The on-line player chooses an on-line algorithm and it is known to the adversary. The adversary then chooses an input sequence. The payoff to the adversary is the ratio of optimal off-line cost to on-line cost, or the performance ratio. Assume that the adversary generates the input sequence oblivious to the on-line player's random choices. With full knowledge of the probability distributions of the on-line algorithm, the input sequence is produced by the oblivious adversary in advance. Therefore, it is straightforward to extend the analysis of the competitive ratio measure to randomized on-line algorithms. Specifically, the randomized competitive ratio with respect to an oblivious adversary is defined the same as (1) with $E[C_A(I)]$ replacing $C_A(I)$ where $E[\cdot]$ is the expected value with respect to the random choices made by A [4,7].

1.2 Related work

In recent years, there has been a large literature on competitive analysis, which is considered as a complementary approach in the analysis of algorithmic decision-making under uncertain conditions. Specifically, the competitive approach is shown to be productive for a variety of on-line financial problems. For the on-line leasing problem, the prototype is the well-known "ski-rental" example put forward by Karp [14] in the field of theoretical computer science. We briefly review the basic leasing model and its main conclusion as follows. In a rental activity, let t be the total number of actual leases, and the costs of renting and purchasing equipment be 1 and a positive integer s, respectively. For the off-line problem, if $s \leq t$, then buy; otherwise rent. For the on-line problem, we consider the following deterministic on-line strategy: rent up to k times and then buy. Thus, if $t \leq k$, then always lease. Denote this on-line strategy by $A(k)$ $(k = 0, 1, 2, \ldots)$, then the optimal on-line strategy is $A(s - 1)$, and its competitive ratio is $2 - 1/s$. A series of research has been carried out on this basic model. Karlin et al. [13] make a significant contribution to on-line analysis for what they call "the ski-rental-family of problem." They give a randomized on-line algorithm with a competitive ratio that is optimal $\frac{e}{e-1} \approx 1.582$. Irani and Ramanathan (1998 Personal Communication) study the situation when the purchase price varies but the rental cost stays fixed, and respectively, give out the upper and lower bounds of the competitive ratio for deterministic and randomized algorithms. They also define a Δ–statistical adversary and present a deterministic algorithm against it. El-Yaniv et al. [7] point out that the investor is often confronted with an important factor—interest rate i, which may be an essential feature of any reasonable financial model. They analyze the leasing problem with the interest rate, and give the competitive ratio of optimal deterministic algorithm $1 + (1 + i)(1 - \frac{1}{s})(1 - s\frac{i}{i+1})$ (therein, if $i = 0$, then

$2-\frac{1}{s}$) and the competitive ratio of optimal randomized algorithm $2-\frac{(\frac{s}{s-1})^{\gamma}-2}{(\frac{s}{s-1})^{\gamma}-1}$, where $\gamma = \frac{\ln(1-s(1-\frac{1}{1+i}))}{\ln\frac{1}{1+i}}$ (therein, if $i \to 0$, then $\gamma = s$; if $s \to \infty$, then $2-\frac{e-2}{e-1} \approx 1.582$). al-Binali [2] builds a famous Risk–Reward framework to analyze the rental problem and the unidirectional trading problem. The framework explains that investors may be willing to increase their risk for some form of reward. Albers et al. [1] introduce and explore natural delayed information and action models to analyze several well-known on-line problems inclusive of a rental problem in which time relevant information is available and the time in which a decision has an effect may be decoupled. Moreover, there are many extensible researches of the basic rental problem [3,6,8].

The previous research always avoided possibility distribution assumption. Raghavan [18], however, proposes a middle ground: the *statistical adversary*—an adversary who generates the input that is constrained to satisfy certain statistical properties. Raghavan uses this notion to analyze a simple version of the *asset allocation problem* in investment theory. Fujiwara and Iwama [10] first break through the customary rule to integrate a possibility distribution assumption into the pure competitive analysis in order to study the on-line leasing problem, where by they assume that the input sequences are subject to an exponential distribution. By integrating more information, they obtain more favorable results that are consistent with reality.

Other than the above-mentioned methods of competitive analysis, there are many other studies of the rental problem. Indeed, it is an optimal stopping problem, so the sequential decision-making method that plays an important role in mathematical statistics can be also used to solve the leasing problem. Moreover, for time sequential decision making to save costs, the investor often emphasizes the time value, and hopes to stop activity immediately if enough information is available. It is beyond the scope of this paper to survey those studies related to the leasing problem. We briefly mention only a few here. For example, [17,19] emphasize asset lease contracts and the tax effects on *lease-and-buy* decision making, and [21,22] discuss a class of Bayesian optimal stopping and a decision rule of geometric distribution.

1.3 Our contribution

The purpose of this study is to improve the performance measurement of competitive analysis by integrating more information of the input structures into pure competitive analysis. We first extend the model in [10] by introducing the interest rate and tax rate to obtain a more realistic solution of the problem. While some of the results in [10] still hold, we find some new results that do not exist in the Fujiwra and Iwama model. Therefore, we generalize the Fujiwra and Iwama results. However, the gain is without a cost. The introduction of new parameters makes the analysis much more complicated.

We find that if the average cost of always leasing without an interest rate is less than the reciprocal of the product between the discount factor and the relative opportunity cost to purchase the equipment, then the optimal strategy for an investor is to lease the equipment forever. Otherwise, the optimal strategy for an investor is to purchase the equipment after leasing for several periods. Moreover, using the forward difference method, we also investigate the discrete leasing model without an interest rate and with an interest rate in different probabilistic environments. The reason that we consider the geometric distribution comes from the *coin-tossing* idea that the leasing does not cease until the purchasing appears. Moreover, in probability theory, the leasing activity in every period is similar to conducting a Bernoulli trial to rent continuously or to purchase immediately. More important, the leasing problem is essentially discrete, so a geometric distribution may be more reasonable to depict its input structures. From numerical analysis, it can be found that the introduction of an interest rate and a tax rate would diminish the uncertainty involved in the process of decision making and the optimal purchasing date would be put off.

It is well known that the worst-case competitive analyses have provided us with many analytically elegant results. However, they have serious limitations [5,9]. The pure competitive analysis always assumes that an on-line player has no information for input sequences. This assumption seems to be unrealistic when a decision-maker does have some partial (statistical) information on the pattern of input sequences [12]. Therefore, the application of competitive algorithms may lead to inferior performance relative to Bayesian algorithms in such cases. Note that the competitive analysis of the worst case emphasizes the difficulty of estimating the input distribution. While it is possible for many combinatorial problems with more complicated input structures [11,23], we can find a number of interesting on-line problems with relatively simple and tractable input structures. We can accurately characterize their input structures by using statistical theory to perform a stochastic competitive analysis. We believe that our analysis, as well as the results in [10], will be helpful in overcoming these difficulties.

The rest of the paper is organized as follows. In Sect. 2, we review the Fujiwra and Iwama model with no interest rate and with the input information drawn from an exponential distribution. we then introduce the interest rate and tax rate into this continuous model to extend the results in [10]. In Sect. 3, because the rental problem is a discrete problem, we build a discrete model with input information drown from a geometric distribution. We first analyze the case without an interest rate and tax rate, and then introduce the two rates into the discrete model and obtain new results. For example, with the introduction of an interest rate and a tax rate, the uncertainty involved in decision making diminishes, and the optimal purchasing date is put off. Moreover, several numerical examples are given in Sect. 4. Finally, Sect. 5 concludes the paper.

2 Optimal analysis of the continuous model

We consider the following deterministic on-line leasing strategy $A(k)$: rent up to k times and then buy. Let $\mathrm{Cost}_{\mathrm{ON}}(t,k)$ and $\mathrm{Cost}_{\mathrm{OPT}}(t)$ denote the cost of the on-line algorithm and the cost of the optimal off-line algorithm, respectively, where t is the total number of the actual leases. Fujiwara and Iwama study the continuous model and propose the stochastic competitive ratio as follows.

Definition 1 Let the total number of the actual leases be a stochastic variable X, which is drawn from a known probability distribution with probability density function $f(t)$. The stochastic competitive ratio is then defined as

$$C(k) = E_X \frac{\mathrm{Cost}_{\mathrm{ON}}(X,k)}{\mathrm{Cost}_{\mathrm{OPT}}(X)} = \int_{t=0}^{\infty} \frac{\mathrm{Cost}_{\mathrm{ON}}(t,k)}{\mathrm{Cost}_{\mathrm{OPT}}(t)} \cdot f(t)\mathrm{d}t. \tag{2}$$

The on-line players use $f(t)$ to estimate the input structures.

For the leasing problem, let $f(t)$ be the exponential distribution function $f(t) = \lambda e^{-\lambda t}$ $(\lambda > 0)$. This distribution means that the hazard rate of immediately purchasing in every period of activity is λ, and the hazard rate of continuously renting in every period of activity is $1 - \lambda$.

2.1 Leasing in a market without an interest rate

Let the costs of renting and purchasing the equipment be 1 and a positive integer s, respectively. For the off-line problem, if $s \le t$, then buy; otherwise rent. The first observation is that

$$\mathrm{Cost}_{\mathrm{OPT}}(t) = \min\{s, t\}. \tag{3}$$

For the on-line problem, if $t \le k$, then always lease. According to on-line strategy $A(k)$ $(k = 0, 1, 2, \ldots)$, then it is not difficult to see that

$$\mathrm{Cost}_{\mathrm{ON}}(t, k) = \begin{cases} t, & t \le k, \\ k + s, & t > k. \end{cases} \tag{4}$$

Obviously, the optimal strategy is immediate purchasing if s is equal to 1, so s is at least 2.

According to Eqs. 2, 3, and 4, Fujiwara and Iwama obtain that, for $0 < k \le s$,

$$C_1(k) = 1 - e^{-\lambda k} + (k + s) \int_k^s \frac{1}{t} \lambda e^{-\lambda t} dt + \frac{k + s}{s} e^{-\lambda s} \tag{5}$$

and for $k > s$,

$$C_2(k) = 1 + \frac{1}{\lambda s} e^{-\lambda s} - \left(\frac{1}{\lambda s} - 1 \right) e^{-\lambda k}. \tag{6}$$

Applying the derivative method, they derive the following results [10].

Theorem 1 *The following strategy provides an optimal stochastic competitive ratio for the exponential input distribution* $f(t) = \lambda e^{-\lambda t}$ $(\lambda > 0)$*: (1) if* $\frac{1}{\lambda} \le s$*, then the investor should rent the equipment forever, and its competitive ratio is* $1 + \frac{1}{\lambda} e^{-\lambda s}$*. (2) If* $\frac{1}{\lambda} > s$*, then the investor should purchase the equipment after renting* k_0 *times, where* k_0 *satisfies* $s^2 \lambda - \frac{s}{10} < k_0 < s^2 \lambda$*, and its competitive ratio is* $1 - \left(1 - \lambda s - \frac{\lambda s^2}{k_0}\right) e^{-\lambda k_0}$.

2.2 Leasing in a market with an interest rate

The net present values of alternative investments are the main concern when making financial decisions in the capital market. The interest rate is obviously an important factor to consider in financial theory [7]. In our leasing problem, we will now consider the effect of a financial market interest rate, defined as i, on the on-line decisions. We assume that the on-line player needs the equipment throughout the entire n contiguous periods. Furthermore, let τ be the tax rate that is a proportion of the purchasing cost of the equipment. In general, it is reasonable to assume that $\frac{1}{s(1+\tau)} > \frac{i}{1+i}$ because the purchase price of the equipment must be less than the present discount value of the alternative of always leasing $\left(s(1 + \tau) < \sum_{j=0}^{\infty} \frac{1}{(1+i)^j}\right)$. Otherwise, by forever leasing the equipment without purchasing, the on-line player can attain a competitive ratio of 1. Define $\beta = \frac{1}{1+i}$, and then $\frac{1}{s(1+\tau)} + \beta - 1 > 0$. This is the relative opportunity cost to purchase the equipment. To make our expression simple, let $\xi = \frac{1}{s(1+\tau)} + \beta - 1$.

Clearly, assume that as $\frac{1}{s(1+\tau)} > \frac{i}{1+i}$, the adversary player will never purchase the equipment after leasing it for some time (as in [7]). Therefore, for any n, the optimal off-line decision-making cost is

$$\text{Cost}_{\text{OPT}}(t) = \begin{cases} \frac{1-\beta^t}{1-\beta}, & t \le n^*, \\ s(1+\tau) & t > n^*, \end{cases} \tag{7}$$

where n^* is the number of rentals of which the total present value is $s(1+\tau)$. In other words, n^* is the root of $\frac{1-\beta^n}{1-\beta} = s(1+\tau)$. That is,

$$n^* = \frac{\ln(1 - s(1+\tau)(1-\beta))}{\ln \beta} = \frac{\ln\left(1 - \frac{is(1+\tau)}{1+i}\right)}{\ln \frac{1}{1+i}}.$$

Based on the strategies, set $A(k)$: rent k times and then buy, where $0 \le k \le n-1$, the on-line decision-making cost is obviously

$$\text{Cost}_{\text{ON}}(t,k) = \begin{cases} \frac{1-\beta^t}{1-\beta}, & t \le k, \\ s\beta^k(1+\tau) + \frac{1-\beta^k}{1-\beta} & t > k. \end{cases} \tag{8}$$

According to (2), (7), and (8), we can obtain that, for $0 < k \le n^*$,

$$C_1(k) = 1 - \mathrm{e}^{-\lambda k} + (1 - \beta^{n^*+k}) \int_k^{n^*} \frac{\lambda \mathrm{e}^{-\lambda t}}{1-\beta^t} \mathrm{d}t + \left[\beta^k + \frac{1-\beta^k}{s(1+\tau)(1-\beta)}\right] \mathrm{e}^{-\lambda n^*} \tag{9}$$

and for $k > n^*$,

$$C_2(k) = 1 - \mathrm{e}^{-\lambda n^*} + \frac{1}{s(1+\tau)(1-\beta)} \int_{n^*}^{k} (1-\beta^k)\lambda \mathrm{e}^{-\lambda t} dt + \left[\beta^k + \frac{1-\beta^k}{s(1+\tau)(1-\beta)}\right] \mathrm{e}^{-\lambda k}. \tag{10}$$

Note that, for $i \to 0$ and $\tau = 0$, it can be shown that $n^* \to s$. The optimal off-line cost (7) and the on-line cost (8) then degenerate into (3) and (4) without an interest rate and a tax rate, respectively. Accordingly, the expressions (9) and (10) degenerate into (5) and (6), respectively. Thus similar to the analysis of Theorem 1, we have the following results.

Theorem 2 *Suppose that the inputs are drawn from the exponential distribution function* $f(t) = \lambda \mathrm{e}^{-\lambda t}$ $(\lambda > 0)$. *Let i be the interest rate in the financial market, and* $\delta(i) = \frac{(1+i)\ln(1+i)}{i}$. *Then, the following strategy provides an optimal stochastic competitive ratio.*

1. *If* $\frac{1}{\lambda} < \frac{1}{\xi \cdot \delta(i)}$, *then the average cost of always leasing without an interest rate is less than the reciprocal of the product between the discount factor and the relative opportunity cost. The optimal strategy for an investor is to lease the equipment forever, and the competitive ratio is* $1 + \frac{\xi \cdot \delta(i)}{\lambda - \ln \beta} \mathrm{e}^{-\lambda n^*}$.

2. *If* $\frac{1}{\lambda} = \frac{1}{\xi \cdot \delta(i)}$*, then the average cost of always leasing without an interest rate is equal to the reciprocal of the product between the discount factor and the relative opportunity cost. The optimal strategy for an investor is to purchase the equipment after* n^* *periods, and the competitive ratio is* $1 + \beta^{n^*} \mathrm{e}^{-\lambda n^*}$.
3. *If* $\frac{1}{\lambda} > \frac{1}{\xi \cdot \delta(i)}$*, where* $\lambda > \frac{s(1+\tau)\ln\beta}{2}(1 - \sqrt{\frac{4}{s(1+\tau)(1-\beta)} - 3})$*, then the average cost of always leasing without an interest rate is more than the reciprocal of the product between the discount factor and the relative opportunity cost. The optimal strategy for an investor is to purchase the equipment after* k_0 *periods, and the competitive ratio is* $1 - [1 - \frac{\lambda s \beta^{k_0}(1+\tau)}{\delta(i)(1-\beta^{k_0})} + \frac{\lambda}{\xi \cdot \delta(i)(1-\beta^{k_0})}]\mathrm{e}^{-\lambda k_0}$*, where* k_0 *satisfies* $k_0 < \frac{1}{\ln\beta}\ln(1 - \frac{\lambda}{\xi \cdot \delta(i)})$*, where* k_0 *is established by using the dichotomous search algorithm in the polynomial time* $O(\log(\frac{1}{\ln\beta}\ln(1 - \frac{\lambda}{\xi \cdot \delta(i)})))$.
4. *If* $\frac{1}{\lambda} \to \infty$*, i.e. the average cost of always leasing without an interest rate approaches* $+\infty$*, then the optimal competitive ratio of any strategy* $A(k)$ *is* $\frac{1}{s(1+\tau)(1-\beta)} + (1 - \frac{1}{s(1+\tau)(1-\beta)})\beta^k$*. The optimal strategy for an investor is to purchase the equipment at the very beginning, and the competitive ratio approaches 1.*

Proof Differentiating $C_1(k)$ and $C_2(k)$, respectively, we can get their first and second-order derivatives.

$$\frac{\mathrm{d}C_1(k)}{\mathrm{d}k} = \lambda\mathrm{e}^{-\lambda k} - \beta^{n^*+k}\ln\beta\int_k^{n^*}\frac{\lambda\mathrm{e}^{-\lambda t}}{1-\beta^t}\mathrm{d}t - (1-\beta^{n^*+k})\frac{\lambda\mathrm{e}^{-\lambda k}}{1-\beta^k} - \frac{\beta^{n^*+k}\ln\beta}{s(1+\tau)(1-\beta)}\mathrm{e}^{-\lambda n^*}$$

$$\frac{\mathrm{d}^2C_1(k)}{\mathrm{d}k^2} = -\lambda^2\mathrm{e}^{-\lambda k} - \beta^{n^*+k}(\ln\beta)^2\int_k^{n^*}\frac{\lambda\mathrm{e}^{-\lambda t}}{1-\beta^t}\mathrm{d}t - 2\beta^{n^*+k}\ln\beta\frac{\lambda\mathrm{e}^{-\lambda k}}{1-\beta^k} - \frac{\beta^{n^*+k}(\ln\beta)^2}{s(1+\tau)(1-\beta)}\mathrm{e}^{-\lambda n^*} - (1-\beta^{n^*+k})\frac{-\lambda^2(1-\beta^k)\mathrm{e}^{-\lambda k} + \lambda\mathrm{e}^{-\lambda k}\ln\beta\beta^k}{(1-\beta^k)^2}$$

$$\frac{\mathrm{d}C_2(k)}{\mathrm{d}k} = \left[(1 - \frac{1}{s(1+\tau)(1-\beta)})\ln\beta - \lambda\right]\beta^k\mathrm{e}^{-\lambda k}.$$

For $(1 - \frac{1}{s(1+\tau)(1-\beta)})\ln\beta < \lambda$, we have

$$\begin{aligned}\frac{\mathrm{d}C_1(k)}{\mathrm{d}k} &< \lambda\mathrm{e}^{-\lambda k} + \lambda s\beta^k(1+\tau)(1-\beta)\int_k^{n^*}\frac{\lambda\mathrm{e}^{-\lambda t}}{1-\beta^t}\mathrm{d}t - \frac{1-\beta^{n+k}}{1-\beta^k}\lambda\mathrm{e}^{-\lambda k} + \lambda\beta^k\mathrm{e}^{-\lambda n^*}\\ &< \lambda\mathrm{e}^{-\lambda k} + \frac{\lambda\beta^k(1-\beta^n)}{1-\beta^k}(\mathrm{e}^{-\lambda k} - \mathrm{e}^{-\lambda n^*}) - \frac{1-\beta^{n+k}}{1-\beta^k}\lambda\mathrm{e}^{-\lambda k} + \lambda\beta^k\mathrm{e}^{-\lambda n^*}\\ &= \frac{\lambda\beta^{2k}\mathrm{e}^{-\lambda n^*}}{1-\beta^k}(\beta^{n^*-k} - 1) < 0.\end{aligned}$$

It is obvious that $C_1(k)$ is a decreasing function of k for $0 < k \le n^*$; therefore $C_1(k)$ becomes minimal when $k = n^*$. Note that $\frac{\mathrm{d}C_2(k)}{\mathrm{d}k} = \left[(1 - \frac{1}{s(1+\tau)(1-\beta)})\ln\beta - \lambda\right]\beta^k\mathrm{e}^{-\lambda k} < 0$. It follows that $C_2(k)$ is a decreasing function of k for $k > n^*$, and $C_2(k)$ becomes minimal when $k \to \infty$. The optimal competitive ratio is $1 + \frac{\xi \cdot \delta(i)}{\ln\beta - \lambda}\mathrm{e}^{-\lambda n^*}$.

For $(1-\frac{1}{s(1+\tau)(1-\beta)})\ln\beta>\lambda$, it is simple to check that

$$\lim_{k\to 0}\frac{\mathrm{d}C_1(k)}{\mathrm{d}k}=-\infty,\quad \lim_{k\to n^*}\frac{\mathrm{d}C_1(k)}{\mathrm{d}k}=\left[\left(1-\frac{1}{s(1+\tau)(1-\beta)}\right)\ln\beta-\lambda\right]\beta^{n^*}\mathrm{e}^{-\lambda n^*}>0.$$

For $\lambda>\frac{s(1+\tau)\ln\beta}{2}(1-\sqrt{\frac{4}{s(1+\tau)(1-\beta)}-3})$, we can prove that $\frac{\mathrm{d}^2C_1(k)}{\mathrm{d}k^2}>0$ as follows.

$$\begin{aligned}\frac{\mathrm{d}^2C_1(k)}{\mathrm{d}k^2}&=\beta^k\left[\frac{1-\beta^{n^*}}{1-\beta^k}\lambda^2\mathrm{e}^{-\lambda k}+\frac{2\beta^{n^*}-\beta^{n^*+k}-1}{(1-\beta^k)^2}\lambda\mathrm{e}^{-\lambda k}\ln\beta\right.\\&\quad\left.-\beta^{n^*}(\ln\beta)^2\int_k^{n^*}\frac{\lambda\mathrm{e}^{-\lambda t}}{1-\beta^t}\mathrm{d}t-\frac{\beta^{n^*}(\ln\beta)^2}{1-\beta^{n^*}}\mathrm{e}^{-\lambda n^*}\right]\\&>\beta^k\left[\frac{1-\beta^{n^*}}{1-\beta^k}\lambda^2\mathrm{e}^{-\lambda k}+\frac{2\beta^{n^*}-\beta^{n^*+k}-1}{(1-\beta^k)^2}\lambda\mathrm{e}^{-\lambda k}\ln\beta\right.\\&\quad\left.-\frac{\beta^{n^*}(\ln\beta)^2}{1-\beta^k}\int_k^{n^*}\lambda\mathrm{e}^{-\lambda t}\mathrm{d}t-\frac{\beta^{n^*}(\ln\beta)^2}{1-\beta^{n^*}}\mathrm{e}^{-\lambda n^*}\right]\\&=\beta^k\left[\frac{1-\beta^{n^*}}{1-\beta^k}\lambda^2\mathrm{e}^{-\lambda k}+\frac{2\beta^{n^*}-\beta^{n^*+k}-1}{(1-\beta^k)^2}\lambda\mathrm{e}^{-\lambda k}\ln\beta-\frac{\beta^{n^*}(\ln\beta)^2}{1-\beta^k}\mathrm{e}^{-\lambda k}\right]\\&\quad+\beta^{n^*+k}(\ln\beta)^2\mathrm{e}^{-\lambda n^*}\left(\frac{1}{1-\beta^k}-\frac{1}{1-\beta^{n^*}}\right)\\&>\beta^k\mathrm{e}^{-\lambda k}\left[\frac{1-\beta^{n^*}}{1-\beta^k}\lambda^2+\frac{2\beta^{n^*}-\beta^{n^*+k}-1}{(1-\beta^k)^2}\ln\beta\cdot\lambda-\frac{\beta^{n^*}(\ln\beta)^2}{1-\beta^k}\right],\end{aligned}$$

where the expression in brackets in the last inequality is a quadratic function of λ. To obtain $\frac{\mathrm{d}^2C_1(k)}{\mathrm{d}k^2}>0$, the lower bound λ is estimated by the following:

$$\begin{aligned}&\frac{1}{2\cdot\frac{1-\beta^{n^*}}{1-\beta^k}}\left[-\frac{2\beta^{n^*}-\beta^{n^*+k}-1}{(1-\beta^k)^2}\ln\beta\right.\\&\qquad\left.+\sqrt{\frac{(\ln\beta)^2(2\beta^{n^*}-\beta^{n^*+k}-1)^2}{(1-\beta^k)^4}+4\cdot\frac{\beta^{n^*}(\ln\beta)^2(1-\beta^{n^*})}{(1-\beta^k)^2}}\right]\\&\quad=\frac{-\ln\beta[2\beta^{n^*}-\beta^{n^*+k}-1+\sqrt{(2\beta^{n^*}-\beta^{n^*+k}-1)^2+4\beta^{n^*}(1-\beta^{n^*})(1-\beta^k)^2}]}{2(1-\beta^{n^*})(1-\beta^k)}\\&\quad<-\frac{\ln\beta}{2(1-\beta^{n^*})(1-\beta^k)}[-(1-\beta^{n^*})^2+\sqrt{(1-\beta^{n^*})^2+4\beta^{n^*}(1-\beta^{n^*})(1-\beta^{n^*})^2}]\\&\quad=\frac{(1-\beta^{n^*})\ln\beta}{2(1-\beta^k)}\left[1-\sqrt{1+\frac{4\beta^{n^*}}{1-\beta^{n^*}}}\right]\\&\quad=\frac{s(1+\tau)\ln\beta}{2(1+\beta+\beta^2+\cdots+\beta^{k-1})}\left(1-\sqrt{\frac{4}{s(1+\tau)(1-\beta)}-3}\right)\\&\quad<\frac{s(1+\tau)\ln\beta}{2}\left(1-\sqrt{\frac{4}{s(1+\tau)(1-\beta)}-3}\right).\end{aligned}$$

For the optimal decision-making date k_0, $\frac{\mathrm{d}^2 C_1(k)}{\mathrm{d}k^2} > 0$ if and only if there is a value k_0. Hence, $C_1(k_0)$ becomes minimal when $0 < k < n^*$. Note that the expression $\frac{\mathrm{d}C_1(k)}{\mathrm{d}k}|_{k=k_0} = 0$ is a non-linear equation on k_0. We have difficulty in obtaining an analytic representation of k_0. However, we obtain the upper bound of k_0 as follows. Applying $\frac{1}{1-\beta^{n^*}} < \frac{1}{1-\beta^k}$ $(k < t < n^*)$ to the first derived term $\frac{\mathrm{d}C_1(k)}{\mathrm{d}k}$, we have

$$\begin{aligned}\frac{\mathrm{d}C_1(k)}{\mathrm{d}k} &> \lambda \mathrm{e}^{-\lambda k} - \frac{\beta^{n^*+k}\ln\beta}{1-\beta^{n^*}}\int_k^{n^*}\lambda \mathrm{e}^{-\lambda t}\mathrm{d}t - \frac{1-\beta^{n^*+k}}{1-\beta^k}\lambda \mathrm{e}^{-\lambda k} - \frac{\beta^{n^*+k}\ln\beta}{1-\beta^{n^*}}\mathrm{e}^{-\lambda n^*}\\ &= \frac{-\beta^k \mathrm{e}^{-\lambda k}}{(1-\beta^k)(1-\beta^{n^*})}[\lambda(1-\beta^{n^*})^2 + (1-\beta^k)\beta^{n^*}\ln\beta].\end{aligned}$$

The last expression becomes zero when $k = \frac{1}{\ln\beta}\ln(1 + \frac{\lambda(1-\beta^{n^*})}{\beta^{n^*}\ln\beta})$, which means that $k_0 < \frac{1}{\ln\beta}\ln(1 - \frac{\lambda}{\xi\cdot\delta(i)})$ holds for the solution k_0 satisfying $\frac{\mathrm{d}C_1(k)}{\mathrm{d}k} = 0$.

If we consider the worst case analysis, i.e. a larger n^*, then the value k_0 may be derived by the dichotomous search algorithm in the polynomial time. Let $B(k) = \frac{\mathrm{d}C_1(k)}{\mathrm{d}k}$ $(k = 1, 2, \ldots, n^*)$. The dichotomous search algorithm to determine the value k_0 in a finite increasing sequence is given below.

Algorithm 1

Step 1 Input $B(1)$, $B(2)$, $B(3)$, $\ldots$, $B(n^*)$.

Step 2 If $B(\lfloor\frac{n^*+1}{2}\rfloor + 1) > 0$, then delete sequence $B(\lfloor\frac{n^*+1}{2}\rfloor + 2)$, $B(\lfloor\frac{n^*+1}{2}\rfloor + 3), \ldots, B(n^*)$; if $B(\lfloor\frac{n^*+1}{2}\rfloor + 1) < 0$, then delete sequence $B(1)$, $B(2), \ldots$, $B(\lfloor\frac{n^*+1}{2}\rfloor)$.

Step 3 Repeat Step 2 until two items remain in the sequence and compare their absolute sizes, and then output the subscript of the minimal absolute value, i.e. k_0.

It is easy to know that its time complexity is $O(\log n^*)$.

If $\frac{1}{\lambda} \to +\infty$, then $\lambda \to 0$. We limit this to the expression $C(k)$. We obtain that the optimal competitive ratio of any strategy $A(k)$ is $\frac{1}{s(1+\tau)(1-\beta)} + (1 - \frac{1}{s(1+\tau)(1-\beta)})\beta^k$, which increases monotonically on k. This is obviously true for $0 < k \leq n^*$. For $n^* < k$, we can conclude that $C_2(k) \to \frac{1}{s(1+\tau)(1-\beta)} + (1 - \frac{1}{s(1+\tau)(1-\beta)})\beta^k$, namely that it converges to a linear function of k. It is obvious for optimal strategy to purchase the equipment at the very beginning, and the competitive ratio is limited to 1.

Note that we can extend the results of Theorem 1. If $i \to 0$ and $\tau \to 0$, then $n^* \to s$, and $\frac{1}{\xi\cdot\delta(i)} \to s$. The competitive ratio of case 1 in Theorem 2 is limited to $1 + \frac{1}{\lambda s}\mathrm{e}^{-\lambda s}$, and the competitive ratio of case 2 in Theorem 2 limits to $1 + \mathrm{e}^{-1}$. Sect. 4 will show by comparing Theorem 2 with Theorem 1 by numerical analysis, that there may be one interesting feature: that with the introduction of the interest rate and tax rate, the uncertainty involved in decision making diminishes, and the optimal purchasing date is put off.

3 Optimal analysis of the discrete model

In actual decision-making activities, many problems are essentially discrete. The input structure is especially simple for the rental problem: its decision-making aim is to seek the optimally critical point. However, in probabilistic and statistical theory, many probabilistic distributions can effectively depict this characteristic. For the discrete problem, the most representative function is the geometric distribution that is often one of the survival functions. For the decision-making problem of on-line leasing, the input sequence shows that the leasing does not cease until purchasing appears, similar to the *Bernoulli* trial of whether to rent continuously or purchase immediately. Thus, its structure has exactly the property of the geometric distribution. In this study it may be more reasonable to depict the input structures for the leasing problem. Here we use the forward difference method to consider the discrete leasing models without an interest rate and with an interest rate. Similarly, the concept of the stochastic competitive ratio can be defined as follows.

Definition 2 Let the number of leases be a stochastic variable X subject to some type of probability distribution function $P(X = t)$. The discrete stochastic competitive ratio is then defined as

$$C(k) = E_X \frac{\text{Cost}_{\text{ON}}(X,k)}{\text{Cost}_{\text{OPT}}(X)} = \sum_{t=0}^{\infty} \frac{\text{Cost}_{\text{ON}}(t,k)}{\text{Cost}_{\text{OPT}}(t)} P(X = t), \tag{11}$$

where $P(X = t)$ is a probability function that is used by the investors to approximate the input structures.

For the leasing problem, we consider the geometric distribution function $P(X = t) = \theta^{t-1}(1-\theta),\ (t = 0, 1, 2, 3, \ldots)$, where θ is the hazard rate of continuous leasing in every period, and $1 - \theta$ is the hazard rate of immediately purchasing in every period.

3.1 Leasing in a market without an interest rate

Let the costs of leasing and purchasing the equipment be 1 and s, respectively. Let τ be the tax rate that is a proportion of the price s. For the off-line problem, if $s(1+\tau) \le t$ then buy, otherwise rent. The first observation is that

$$\text{Cost}_{\text{OPT}}(t) = \min\{s(1+\tau), t\}. \tag{12}$$

For the on-line problem, if $t \le k$, then always lease. According to the on-line strategy $A(k)$ $(k = 0, 1, 2, \ldots)$, then

$$\text{Cost}_{\text{ON}}(t,k) = \begin{cases} t, & t \le k, \\ k + s(1+\tau) & t > k. \end{cases} \tag{13}$$

Obviously, the optimal strategy is to immediately purchase if $s(1+\tau)$ is equal to 1, so $s(1+\tau)$ is at least 2.

According to Eqs. 11, 12, and 13, we have, for $k = 0, 1, 2, 3, \ldots, (1+\tau)s$, that

$$C(k) = (1-\theta^k) + (k + s(1+\tau))(1-\theta) \sum_{t=k+1}^{s(1+\tau)} \frac{1}{t}\theta^{t-1} + \frac{k + s(1+\tau)}{s(1+\tau)}\theta^{s(1+\tau)} \tag{14}$$

and for $k = s(1+\tau)+1, s(1+\tau)+2, s(1+\tau)+3, \ldots,$

$$C(k) = (1-\theta^{s(1+\tau)}) + \frac{1-\theta}{s(1+\tau)} \sum_{t=s(1+\tau)+1}^{k} t\theta^{t-1} + \frac{k + s(1+\tau)}{s(1+\tau)}\theta^k. \tag{15}$$

Then we obtain the following result.

Theorem 3 *Let X be a random variable for the total number of times that the investor leases the equipment, and the inputs are drawn from a geometric distribution that the probability function is $P(X = t) = (1-\theta)\theta^{t-1}$. The following strategy provides an optimal stochastic competitive ratio.*

1. *If $\frac{1}{1-\theta} < s(1+\tau)$, then the average cost of always leasing is less than the purchasing cost $s(1+\tau)$. The optimal strategy for an investor is to lease the equipment forever, and the competitive ratio is $1 + \frac{\theta^{s(1+\tau)}}{s(1+\tau)(1-\theta)}$.*
2. *If $\frac{1}{1-\theta} = s(1+\tau)$, then the average cost of always leasing is equal to the purchasing cost $s(1+\tau)$. The optimal strategy for an investor is to purchase the equipment after leasing for $s(1+\tau)-1$ periods, and the competitive ratio is $1 + (1 - \frac{1}{s(1+\tau)})^{s(1+\tau)}$.*
3. *If $\frac{1}{1-\theta} > s(1+\tau)$, then the average cost of always leasing is greater than the purchasing cost $s(1+\tau)$. The optimal strategy for an investor is to purchase the equipment after leasing for k_0 periods, and the competitive ratio is $1 - [1 - \frac{k_0 s(1+\tau)(1-\theta)}{k_0+1} - \frac{s^2(1-\theta)(1+\tau)^2}{k_0+1}]\theta^{k_0}$, where k_0 satisfies $s^2(1-\theta)(1+\tau)^2 - 0.09s(1+\tau) - 1 < k_0 < s^2(1-\theta)(1+\tau)^2 - 1$. Note that k_0 could be also determined by using the dichotomous search algorithm in the polynomial time $O(\log s(1+\tau))$.*
4. *If $\frac{1}{1-\theta} \to \infty$, then the average cost of always leasing approaches ∞, and the optimal competitive ratio of any strategy $A(k)$ is $1 + \frac{k}{s(1+\tau)}$. The optimal strategy for an investor is to purchase equipment at the very beginning, and the competitive ratio approaches 1.*

Proof For $k = 0$, $A(0)$ may be also an optimal investment strategy in which an investor chooses to buy at the beginning. $A(0)$ often exists in practice, and the competitive ratio is a finite value $C(0) = s(1+\tau)(1-\theta)\sum_{t=1}^{s(1+\tau)} \frac{1}{t}\theta^{t-1} + \theta^{s(1+\tau)}$, where is different from the case in [10] that $C(0) \to \infty$ if $k \to 0$. We discuss the following cases.

Case 1 For $k = 0, 1, 2, \ldots, s(1+\tau) - 2$, we have

$$\begin{aligned} C(k+1) - C(k) &= -\frac{s(1+\tau)(1-\theta)}{k+1}\theta^k + \frac{1}{s(1+\tau)}\theta^{s(1+\tau)} + (1-\theta)\sum_{t=k+1}^{s(1+\tau)} \frac{1}{t}\theta^{t-1} \\ &\le -\frac{s(1+\tau)(1-\theta)}{k+1}\theta^k + \frac{1}{s(1+\tau)}\theta^{s(1+\tau)} + \frac{1-\theta}{k+1} \cdot \frac{\theta^k - \theta^{s(1+\tau)}}{1-\theta} \\ &= \theta^k\left(\frac{1}{k+1} - \frac{s(1+\tau)(1-\theta)}{k+1}\right) + \theta^{s(1+\tau)}\left(\frac{1}{s(1+\tau)} - \frac{1}{k+1}\right) < 0. \end{aligned}$$

For $k = s(1+\tau) - 1$, we note that $C(s(1+\tau)) - C(s(1+\tau)-1) = \left[\frac{1}{s(1+\tau)} - (1-\theta)\right]\theta^{s(1+\tau)-1} < 0$.

For $k = s(1+\tau)+1, s(1+\tau)+2, s(1+\tau)+3, \ldots$, we also have $C(k+1) - C(k) = \left[\frac{1}{s(1+\tau)} - (1-\theta)\right]\theta^k < 0$. It is clear that $C(s(1+\tau)+1) - C(s(1+\tau)) = \left[\frac{1}{s(1+\tau)} - (1-\theta)\right]\theta^{s(1+\tau)} < 0$. Hence, it follows that $C(0) > C(1) > \cdots > C(s(1+\tau)-1) > C(s(1+\tau)) > C(s(1+\tau)+1) > C(s(1+\tau)+2) > \cdots$.

This shows that if $\frac{1}{1-\theta} < s(1+\tau)$, i.e., the average cost of always leasing is less than the purchase cost s, then the optimal strategy for an investor is to lease the equipment forever. Therefore, the competitive ratio $C(k)$ becomes minimum as $k \to \infty$. We can derive the optimal competitive ratio as $1 + \frac{\theta^{s(1+\tau)}}{s(1+\tau)(1-\theta)}$.

Case 2 Similar to the proof of *Case 1*, for $k = 0, 1, 2, 3, \ldots, s(1+\tau) - 2$, we also obtain that $C(k+1) - C(k) < 0$. For $k = s(1+\tau) - 1, s(1+\tau), s(1+\tau) + 1, s(1+\tau) + 2, s(1+\tau) + 3, \ldots$, note that $C(k+1) - C(k) = 0$. Hence, it follows that $C(0) > C(1) > \cdots > C(s(1+\tau)-1) = C(s(1+\tau)) = C(s(1+\tau)+1) = C(s(1+\tau)+2) = \cdots$.

This shows that if $\frac{1}{1-\theta} = s(1+\tau)$, i.e., the average cost of always leasing is equal to the purchase cost $s(1+\tau)$, then their competitive ratios are all equal if the investor buys the equipment at any arbitrary time after $s(1+\tau)-1$ periods, while in the actual investment the optimal decision should be to purchase at period $s(1+\tau)-1$, and the optimal competitive ratio is $1 + (1 - \frac{1}{s(1+\tau)})^{s(1+\tau)}$.

Case 3 Similar to the proof of *Case* 1 and *Case* 2, we can also conclude that $C(s(1+\tau)-1) < C(s(1+\tau)) < C(s(1+\tau)+1) < C(s(1+\tau)+2) < \cdots$. Check that the second order difference $C(k+2)-2C(k+1)+C(k) > 0$ $(k = 0, 1, 2, \ldots, s(1+\tau)-2)$ as follows. Because the inequality $\frac{1}{1-\theta} > s(1+\tau) \geq 2$ can be written as $0 > s(1+\tau)(1-\theta) - 1 \geq 1 - 2\theta$, we obtain $\theta > \frac{1}{2}$, and

$$\begin{aligned} C(k+2) - 2C(k+1) + C(k) &= \theta^k(1-\theta)\left(\frac{-\theta s(1+\tau)}{k+2} + \frac{s(1+\tau)-1}{k+1}\right) \\ &> \theta^k(1-\theta)\frac{(s(1+\tau)-1) + (k+1)(1-2\theta)}{(k+2)(k+1)} > \theta^k(1-\theta)\frac{(k+s(1+\tau)) - 2(k+1)}{(k+2)(k+1)} \\ &\geq 0. \end{aligned}$$

Hence, for $k = 0, 1, 2, \ldots, s(1+\tau)$, there is only a value k_0 such that $C(k_0)$ becomes minimum. We have the following two inequalities, i.e. $C(k_0 - 1) > C(k_0)$ and $C(k_0) < C(k_0+1)$, as follows.

$$\begin{cases} \theta^{k_0-1}(1-\theta)\frac{s(1+\tau)}{k_0} - \frac{1}{s(1+\tau)}\theta^{s(1+\tau)} - (1-\theta)\sum\limits_{t=k_0}^{s(1+\tau)} \frac{1}{t}\theta^{t-1} > 0 \\ \theta^{k_0}(1-\theta)\frac{1-s(1+\tau)}{k_0+1} + \frac{1}{s(1+\tau)}\theta^{s(1+\tau)} + (1-\theta)\sum\limits_{t=k_0+2}^{s(1+\tau)} \frac{1}{t}\theta^{t-1} > 0. \end{cases}$$

As the above inequality group is non-linear with respect to k_0, it is difficult to derive an analytic representation of k_0. However, the upper and lower bounds of k_0 satisfy the following inequality similar to the result in [10]:

$$s(1+\tau) - \frac{\ln(\theta - s(1+\tau)(1-\theta) + s^2(1+\tau)^2(1-\theta)^2)}{\ln\theta} < k_0 < s^2(1-\theta)(1+\tau)^2 - 1.$$

For the upper bound, applying $\frac{1}{t} > \frac{1}{s(1+\tau)}$ $(s(1+\tau) \leq t \leq k)$ to the first order difference $C(k+1) - C(k)$, we obtain

$$\begin{aligned} C(k+1) - C(k) &> -\frac{s(1+\tau)(1-\theta)}{k+1}\theta^k + \frac{1}{s(1+\tau)}\theta^{s(1+\tau)} + \frac{1-\theta}{k+1}\sum_{t=k+1}^{s(1+\tau)}\theta^{t-1} \\ &= \left[\frac{1}{s(1+\tau)} - \frac{s(1+\tau)(1-\theta)}{k+1}\right]\theta^k. \end{aligned}$$

The last expression becomes zero when $k = s^2(1-\theta)(1+\tau)^2 - 1$, which means that $k_0 < s^2(1-\theta)(1+\tau)^2 - 1$ holds for k_0 satisfying $C(k+1) - C(k) = 0$, because the first order difference $C(k+1) - C(k)$ increases monotonically as mentioned earlier.

For the lower bound, using $\frac{1}{t} \leq -\frac{1}{s(1+\tau)(k+1)}(t-s(1+\tau))+\frac{1}{s(1+\tau)}$ $(k \leq t \leq s(1+\tau))$, we obtain

$$C(k+1)-C(k) < -\frac{s(1+\tau)(1-\theta)}{k+1}\theta^k + \frac{1}{s(1+\tau)}\theta^{s(1+\tau)}$$
$$+(1-\theta)\sum_{t=k+1}^{s(1+\tau)}\left[-\frac{t-s(1+\tau)}{s(1+\tau)(k+1)}+\frac{1}{s(1+\tau)}\right]\theta^{t-1}$$
$$=\left[s(1+\tau)(1-\theta)-s^2(1+\tau)^2(1-\theta)^2-\theta+\theta^{s(1+\tau)-k}\right]\frac{\theta^k}{s(1+\tau)(1-\theta)(k+1)}.$$

The last expression yields zero when $k = s(1+\tau) - \frac{\ln(\theta - s(1+\tau)(1-\theta)+s^2(1+\tau)^2(1-\theta)^2)}{\ln\theta}$. Thus, it follows that $s(1+\tau) - \frac{\ln(\theta - s(1+\tau)(1-\theta)+s^2(1+\tau)^2(1-\theta)^2)}{\ln\theta} < k_0$ for the same reason discussed earlier.

We can verify that the difference of the upper and lower bounds is less than $0.09s(1+\tau)$ by numerical simulation. If we consider the worst case analysis, i.e. a large $s(1+\tau)$, let $B(k) = C(k+1)-C(k)$, and then we know that $B(k)$ $(k = 1, 2, \ldots, s(1+\tau))$ is strictly monotone increasing. Similar to Theorem 2, the dichotomous search algorithm is also used to determine the optimal decision-making date k_0 in a finite increasing sequence. It is easy to know that the time complexity of this method is $O(\log s(1+\tau))$. Moreover, we also obtain important information that the hazard rate θ to continue leasing every period is greater than $\frac{1}{2}$. Compared to the decision-making date in [14,7], the optimal purchasing date is advanced and the competitive ratio is $1-[1-\frac{k_0 s(1+\tau)(1-\theta)}{k_0+1} - \frac{s^2(1-\theta)(1+\tau)^2}{k_0+1}]\theta^{k_0}$.

Case 4 If $\frac{1}{1-\theta} \to +\infty$, then $C(k) \to 1 + \frac{k}{s(1+\tau)}$ for the entire range of k.

Whichever case we consider, the competitive ratio that an investor will take the strategy $A(s(1+\tau)-1)$, even when there is a large deviation for the hazard ratio θ from estimation, is obviously better than the deterministic competitive ratio $2-\frac{1}{s}$ in [14] and the randomized competitive ratio in [7]. For example, if $s = 10$, $\tau = 0$, and $\theta = 0.95$, then the competitive ratio of 1.56722 in our models is better than the competitive ratio of 1.9 in [14], and the randomized competitive ratio of 1.582 in [7].

3.2 Leasing in a market with an interest rate

According to (11), (7), and (8), we can obtain, for $k = 0, 1, 2, 3, \ldots, n^*$, that

$$C(k) = (1-\theta^k)+(1-\theta)(1-\beta^{n^*+k})\sum_{t=k+1}^{n^*}\frac{1}{1-\beta^t}\theta^{t-1}+(\beta^k+\frac{1-\beta^k}{s(1+\tau)(1-\beta)})\theta^{n^*} \tag{16}$$

and for $k = n^*+1, n^*+2, n^*+3, \ldots,$

$$C(k) = (1-\theta^{n^*})+\frac{1-\theta}{s(1+\tau)(1-\beta)}\sum_{t=n^*+1}^{k}(1-\beta^t)\theta^{t-1}+(\beta^k+\frac{1-\beta^k}{s(1+\tau)(1-\beta)})\theta^k. \tag{17}$$

Note that, for $i \to 0$, we have $n^* \to s(1+\tau)$. The optimal off-line cost (7) and on-line cost (8) degenerates to (12) and (13) without an interest rate and a tax rate, respectively. Accordingly, the expressions (16) and (17) degenerate to (14) and (15), respectively. Thus, similar to the analysis of Theorem 3, we give the following results. For simplicity, we omit the proof here.[1]

Theorem 4 *Let X be a random variable for the total number of times that the investors leases the equipment. Let the input sequence be drawn from a geometric distribution that the probability function is $P(X = t) = \theta^{t-1}(1-\theta)$. Let i be the interest rate in the financial market. The following strategy provides an optimal stochastic competitive ratio.*

1. *If $\frac{1}{1-\theta} < \frac{1}{\xi(1+i)}$, then the average cost of always leasing without an interest rate is less than the present discount value of the reciprocal of the relative opportunity cost. The optimal strategy for an investor is to lease the equipment forever, and the competitive ratio is $1 + \frac{1+\beta(1-2\theta)}{(1-\beta)(1-\beta\theta)}\xi\theta^{n^*}$.*
2. *If $\frac{1}{1-\theta} = \frac{1}{\xi(1+i)}$, then the average cost of always leasing without an interest rate is equal to the present discount value of the reciprocal of the relative opportunity cost. The optimal strategy for an investor is to buy the equipment after n^*-1 periods, and the competitive ratio is $1 + (\frac{\theta}{1+i})^{n^*}$.*
3. *If $\frac{1}{1-\theta} > \frac{1}{\xi(1+i)}$, then the average cost of always leasing without an interest rate is greater than the present discount value of the reciprocal of the relative opportunity cost. The optimal strategy for an investor is to buy the equipment after k_0 periods, and the competitive ratio is $1+(\frac{\beta s(1+\tau)(1-\theta)(1-\beta^{n^*+k_0})}{\beta^{n^*}(1-\beta^{k_0+1})}-1)\theta^{k_0}$, where the decision-making date k_0 is established by using the dichotomous search algorithm in the polynomial time $O(\log n^*)$.*
4. *If $\frac{1}{1-\theta} \to \infty$, i.e. the average cost of always leasing without interest rate approaches $+\infty$, then the optimal competitive ratio of any strategy $A(k)$ is $\frac{1}{s(1+\tau)(1-\beta)} + (1 - \frac{1}{s(1+\tau)(1-\beta)})\beta^k$. The optimal strategy for an investor is to purchase the equipment at the very beginning, and the competitive ratio is limited to 1.*

Note that Theorem 4 is an extension of Theorem 3. If $i \to 0$, then $n^* \to s$, and $\frac{1}{\xi(1+i)} \to s(1+\tau)$. Comparing Theorem 4 to Theorem 3 using numerical analysis in a way that is similar to that in Sect. 2, we find that an interesting feature may be that with the introduction of an interest rate and a tax rate, the uncertainty involved in decision making diminishes, and the optimal purchasing date is put off.

4 Numerical analysis

In this section, we present several numerical examples to develop a more intuitive understanding about our analysis and the effect on the competitive ratio when the parameters in our model are varying. To save space, we only analyze Cases 1 and 3 in our theorems (the analyses of Cases 2 and 4 are obvious). The numerical results are presented in Figs. 1–6 in the appendix. Suppose that there is a company in need of equipment. In the capital market, let the rental cost of the equipment in every period

[1] The proof is available to interested readers upon request.

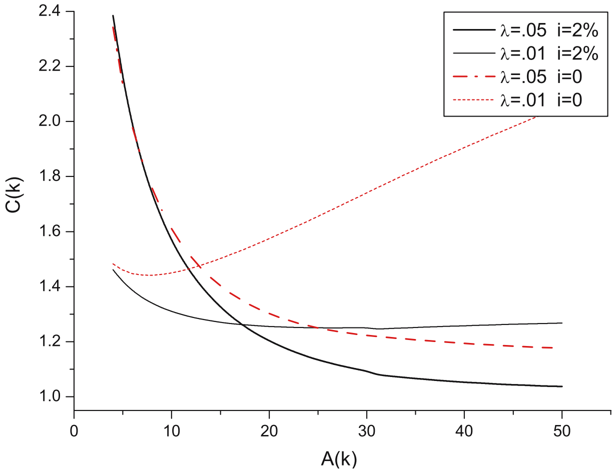

Fig. 1 Case 1 and Case 3 of $r = 1$, $s = 30$, and $\tau = 0$ in Theorem 1 and Theorem 2. With the introduction of the interest rate, the curves of the competitive ratio shift downward and the optimal purchasing date is put off

be a constant 1, and let the purchase price of the equipment be s. From Figs. 1–6, the horizontal coordinates measure the different strategies that the on-line investor takes. The vertical coordinates show the size of the competitive ratio in the different strategies. The curves in the figures show that when the interest rate i and the hazard parameter λ or θ vary, the competitive ratios also vary according to the different strategies taken by the investor. It is easy for us to determine the rental cost in every period and the purchase price. For the estimation of the hazard parameters λ and θ, because the activity has taken place until the moment that the investor buys the equipment after renting it for some periods, the parameters λ and θ can be estimated by using the maximum likelihood method in statistical theory. The samples of the parameters could be obtained according to the information of other investors or the advice of experts. In our analysis, the samples of these parameters can be derived by using computers to simulate experts opinion, and then we use the maximum likelihood method to estimate the value of the parameters. Moreover, we find that different expert groups ultimately lead to different investment strategies. Now, we will further analyze the changes of the competitive ratio when an interest rate is introduced. We present the results in the appendix.

In Fig. 1 in the appendix, when there is no interest rate i in the market, the two curves of $\lambda = 0.05$ and $\lambda = 0.01$ are the cases in Theorem 1: i.e., they are the cases of the Fujiwara and Iwama model [10]. When the interest rate is introduced, we find that the curves shift downward accordingly. This shows that the market interest rate has important effects on the investment decision. The greater the time value of money, the more rational and prudent will be the investor in choosing the investment strategies. Hence, with the existence of an interest rate, the uncertainty involved in (financial) decision making diminishes. Therein, when $i = 2\%$ and $\lambda = 0.05$, the curve of the competitive ratio shifts down monotonously. The lower curve indicates that the average cost of always leasing without an interest rate is less than the present discount value of the reciprocal of the relative opportunity cost. The optimal strategy for the investor is to lease the equipment forever. We also find that the values of the competitive ratio reduce rapidly and then gradually approach the limit value when k varies from 0 to infinity. For example, when an investor takes strategies $A(50)$ and $A(500)$ the values of the competitive ratios are almost the same. When the interest rate $i = 2\%$ and $\lambda = 0.01$, the curve of the competitive ratio first declines and then increases. This patten shows that an investor considers the average cost of always leasing without an interest rate to be greater than the present discount value of the reciprocal of the relative opportunity cost. The optimal strategy for the investor is to buy the equipment after renting for some periods. This explains why there are different types of investors in reality. Some may like to use the rental strategy forever, while others may purchase the equipment after renting for some periods when the competitive ratios hardly change because k is very large. From the theoretical analysis, the optimal strategy should be used when its competitive ratio is minimum. Similar arguments can be made about the other strategy in Fig. 1 and all of the curves in Fig. 4 in the appendix.

In Figs. 2 and 5 in the appendix, we note that the curves of the competitive ratio shift downward with the increase of the interest rate, and both purchasing dates are put off: that is to say, the higher the interest rate, the more level-headed and prudent is the investor when they choose the investment strategy due to the time value of money. In Figs. 3 and 6 in the appendix, we also find that the estimation results of hazard rates λ and θ affect the competitive ratio. Similarly, when the tax rate is introduced, we also obtained numerical results. For example, when the tax rate τ is introduced, the curves of the competitive ratios shift downward accordingly, and the optimal purchasing date is put off. This shows that the tax rate also has important effects on the investment strategy that the investor uses. Moreover, the higher the tax rate, the more that the curves of the competitive ratio shift downward and the optimal purchasing date is put off. This shows that the investor more rationally chooses the investment strategy due to the increase of the tax cost. By the above analysis, with the introduction of some financial factors, the uncertainty involved in decision making diminishes.

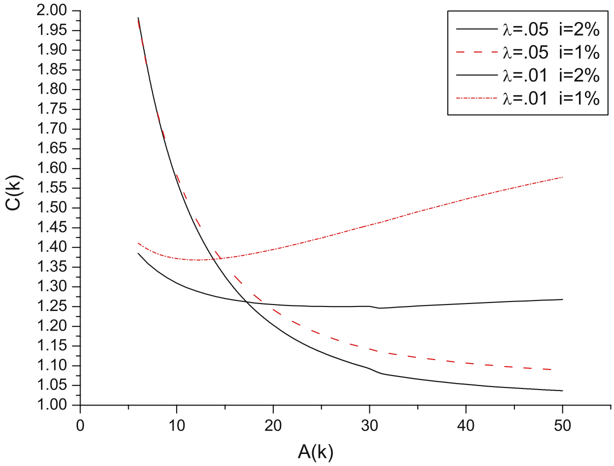

Fig. 2 Case 1 and Case 3 of $r = 1$, $s = 30$, and $\tau = 0$ in Theorem 2. The higher the interest rate, the more the curves of the competitive ratio shift downward and the optimal purchasing date is put off

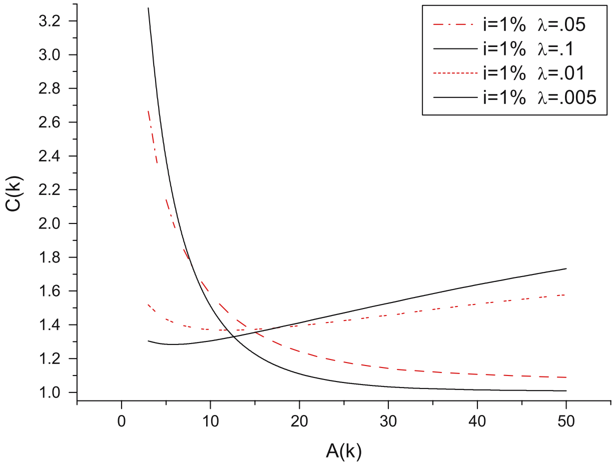

Fig. 3 Case 1 and Case 3 of $r = 1, s = 30$, and $\tau = 0$ in Theorem 2. The changes of parameter λ have significant effects on the competitive ratios

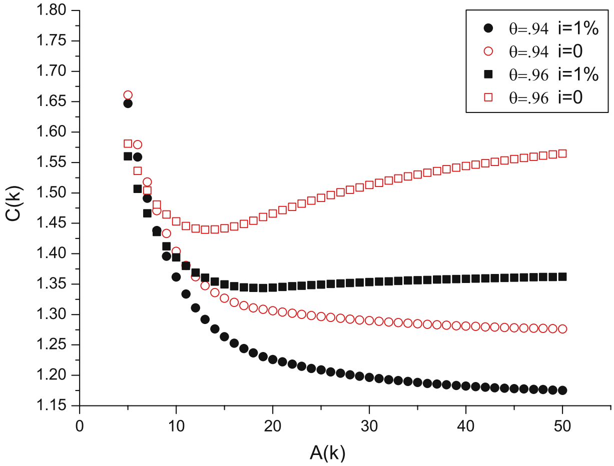

Fig. 4 Case 1 and Case 3 of $r = 1$, $s = 19$, and $\tau = 0$ in Theorem 3 and Theorem 4. With the introduction of the interest rate, the curves of the competitive ratio shift downward and the optimal purchasing date is put off

In the traditional decision making of financial leasing, the approach used is often cost analysis, which is simple and exercisable. Namely, by comparing the rental cost with the purchase cost, we choose the low cost project, which is the optimal strategy. It is exactly the off-line optimal algorithm in the competitive analysis. For the on-line decision problem, because the future demand is entirely uncertain, it is impossible for us to use the traditional method of cost analysis to determine investment decisions. Hence, how to skillfully design the online strategy is very important, because online decisions can offer decision makers a new investment idealogy and inspiration.

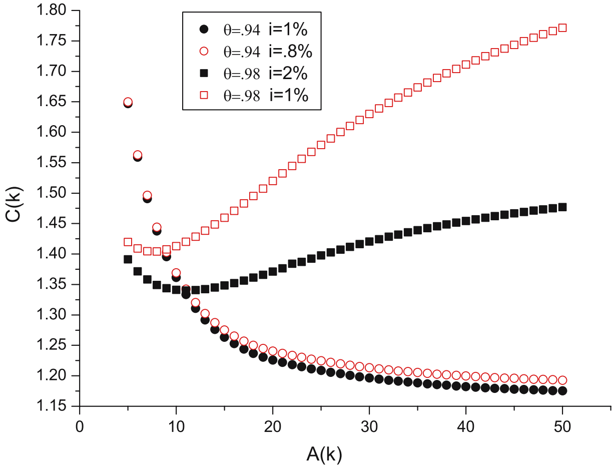

Fig. 5 Case 1 and Case 3 of $r = 1$, $s = 19$, and $\tau = 0$ in Theorem 4. The higher the interest rate, the more the curves of the competitive ratio shift downward and the optimal purchasing date is put off

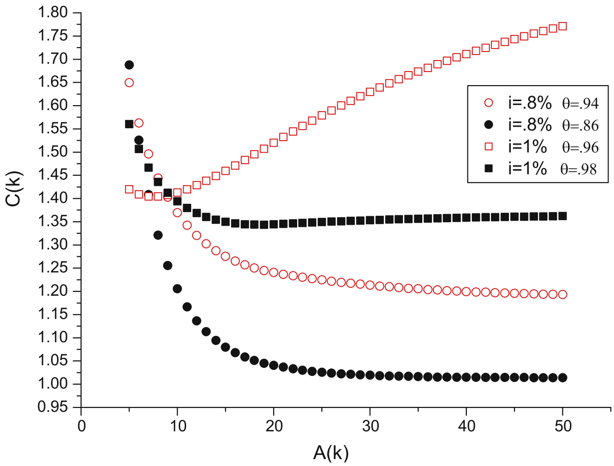

Fig. 6 Case 1 and Case 3 of $r = 1$, $s = 19$, and $\tau = 0$ in Theorem 4. The changes of parameter i have significant effects on the competitive ratios

5 Conclusions

The concept of the competitive ratio is not a new-found and optimal decision-making criterion from a decision-making theory point of view. Although Ran El-Yaniv consummated the axiom system of the competitive ratio, it still has inherent and insurmountable limitations [5]. Hence, a central issue is still how to perfect and improve the analysis performance of the competitive ratio by combining it with other methods. How to depict input information under uncertain conditions is also a focal point, and we think that Rough set theory and possibility distribution could be integrated into pure competitive analysis to improve the performance measures of on-line algorithms. Moreover, as shown in our analysis, we know that the competitive ratio can be improved when more realistic factors are considered. For example, the model of on-line leasing in this paper can be further studied when factors such as inflation, salvage cost, etc. are considered, or when the rental or purchase price fluctuate.

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