Handbook of Combinatorial Optimization

Volume 2

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Preface

Combinatorial (or discrete) optimization is one of the most active fields in the interface of operations research, computer science, and applied mathematics. Combinatorial optimization problems arise in various applications, including communications network design, VLSI design, machine vision, airline crew scheduling, corporate planning, computer-aided design and manufacturing, database query design, cellular telephone frequency assignment, constraint directed reasoning, and computational biology. Furthermore, combinatorial optimization problems occur in many diverse areas such as linear and integer programming, graph theory, artificial intelligence, and number theory. All these problems, when formulated mathematically as the minimization or maximization of a certain function defined on some domain, have a commonality of discreteness.

Historically, combinatorial optimization starts with linear programming. Linear programming has an entire range of important applications including production planning and distribution, personnel assignment, finance, allocation of economic resources, circuit simulation, and control systems. Leonid Kantorovich and Tjalling Koopmans received the Nobel Prize (1975) for their work on the optimal allocation of resources. Two important discoveries, the ellipsoid method (1979) and interior point approaches (1984) both provide polynomial time algorithms for linear programming. These algorithms have had a profound effect in combinatorial optimization. Many polynomial-time solvable combinatorial optimization problems are special cases of linear programming (e.g. matching and maximum flow). In addition, linear programming relaxations are often the basis for many approximation algorithms for solving NP-hard problems (e.g. dual heuristics).

Two other developments with a great effect on combinatorial optimization are the design of efficient integer programming software and the availability of parallel computers. In the last decade, the use of integer programming models has changed and increased dramatically. Two decades ago, only problems with up to 100 integer variables could be solved in a computer. Today we can solve problems to optimality with thousands of integer variables. Furthermore, we can compute provably good approximate solutions to problems with millions of integer variables. These advances have been made possible by developments in hardware, software and algorithm design.

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Minimum Weight Triangulations

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Contents

1 Introduction

A *triangulation* of a given set S of n points in the plane is a maximal set of non-crossing line segments (called *edges*) which have both endpoints in S. A triangulation partitions the interior of the convex hull of the given point set into triangles. It is used in many areas of engineering and scientific applications such as finite element methods, approximation theory, numerical computation, computer-aided geometric design, and etc.

Let $CH(S)$ denote the set of edges bounding the convex hull of S, $T(S)$ denote a triangulation of S. The Eulerian relation for planar graphs implies the following equalities.

$$\begin{aligned} |T(S)| &= 3n - 3 - |CH(S)| \\ |T_t(S)| &= 2n - 2 - |CH(S)| \end{aligned}$$

where $|T(S)|$ is the edge number in $T(S)$ and $|T_t(S)|$ is the triangle number in $T(S)$ [18].

For a given n points set S in the plane, the number of different triangulations of S is a exponential function of n, the best known lower and upper bounds are $\Omega(2^{3n})$ [23] and $\Omega(2^{8n})$ [15]. This fact makes to compute optimal triangulations with some optimality critria become a challenge problem.

Since a triangulation contains edge set and some region set, this enables the multiple applications of different type of triangulations.

Optimization criteria for which efficient algorithms are known include maximizing the minimum angle [18], minimizing the maximum angle [20], minimizing the minimum angle [22], minimizing the maximum aspect ratio [44], and minimizing the maximum edge length [19]. *Delaunay triangulation* which is a dual of *Voronoi diagram* is a maxmin angle triangulation and can be computed in $O(n \log n)$ and $O(n)$ [18, 44] and some other optimality properties in high dimension related with *Delaunay triangulation* were observed by Rajan in [44].

Edelsbrunner and Tan gave an $O(n^2 \log n)$ time algorithm for computing the minmax length triangulation [19]. Edelsbrunner et al.[20] gave an $O(n^2)$ time algorithm to compute a triangulation which minimizes the maximum angle. The former algorithm is based on an *edge insertion paradigm* which is shown in Bern et al.[7] to lead polynomial time algorithms to triangulations with maxmin triange height, minmax triangle eccentricity, and minmax gradient surface, respectively.

For each edge of a triangulation, assign a weight for the edge as the *Euclidean distance* between its two endpoints. Then each triangulation has a weight as the sum of the edge weight in the triangulation. Among all the optimality critria on triangulations, the most outstanding one is to minimize the weight, which is known as the minimum weight triangulation.

The minimum weight triangulation has applications in the numerical approximation of bivariate data [52]. The complexity of computing the minimum weight triangulation for arbitrary planar point set is still open since 1975 when it was mentioned by Shamos and Hoey [45]. It is one of the unsolved open problems listed in Garey and Johnson's book [24]. Since then several algorithms have been proposed to solve this problem [5, 25, 27, 33, 35, 38, 43, 45, 47]. None of these is known to produce the exact minimum weight triangulation in polynomial time.

Finding a minimum length weight triangulation with additional points admit, which is call *Minimum Weight Steiner Triangulation* are also considered and some approximation algorithms were given by Eppstein [21].

In the following, we will focus the topic mainly on *Minimum Weight Triangulation*(denoted by MWT) and refer it as the optimal one.

Attempts to prove the minimum weight triangulation problem *NP-hard* have resulted in two related *NP-hardness* results. Lloyd [40] showed that given a set S of points in the plane and a set of edges with endpoints in S, the problem of determining whether this edge set contains a triangulation is *NP-complete*. Lingas [39] showed that the problem of determining the minimum weight geometric triangulation of multi-connected polygons is *NP-complete*. Another related result was given by Jansen, it is showed that the min-max degree triangulation problem is *NP-complete*[30].

2 Dynamic Programming

2.1 MWT for a Simple Polygon

Dynamic programming is a powerful tool to deal with some discrete optimization problems. Gilbert[25] and Klincsek[33] gave a dynamic programming algorithm for computing a minimum weight triangulation of a convex polygon and also hold for simple polygon. The *time complexity* and the *space complexity* are $O(n^3)$ and $O(n^2)$ respectively. The algorithm can be mentioned as follows.

Let $x_0, x_1, ..., x_{n-1}$ be the n vertices of a convex polygon in clockwise order. Let $C[p_i, s]$, where $i = 0, 1, ..., n-1$ and $s = 0, 1, ..., n$, denote the the sum of weights of the interior edges of the minimum weight triangulation of the convex polygon with vertices $\{x_i, x_{i+1}, ..., x_{i+s-1}\}$, where the subscripts are taken modulo n. Then to find the minimum weight triangulation of the polygon is to determine $C[p_0, n]$. Let $C[p_i, s] = 0, i = 0, 1, ..., n-1$, $1 \le s \le 3$ and

$$C[p_i, s] = \min_{(i+1) \le j \le (i+s-2)} \{w(p_i, p_j) + w(p_j, p_{(i+s-1)}) + C[p_i, j-i+1] + C[p_j, i+s-j]\}$$

where $w(p_i, p_j)$ denotes the Euclidean distance between p_i and p_j and $w(p_i, p_j) = 0$ if the edge with endpoints p_i and p_j is on the boundary of the polygon.

The above dynamic programming shows the following theorem hold.

Theorem 2.1 *A minimum weight triangulation of a simple polygon can be found in $O(n^3)$ time and $O(n^2)$ space.*

For some special points arrangement, the minimum weight triangulation of the point set can also be computed in polynomial time with Gilbert and Klincsek's algorithm. In [47], the minimum weight triangulation with convex layers constraint is studied(the convex layers of a point set S is the set of nested convex polygons obtained by repeatedly computing the convex hull of the remaining set after removing the vertices of the current convex hull.) and it is pointed out that if the convex layers is seperated enough, then the minimum weight triangulation can be computed in $O(n^4)$ time with the above dynamic programming .

With a suitable adaptation of the *Hu-Tucker algorithm* for *optimal alphabetic tree*[28, 29], consider finding an MWT of points on a line, which is a degenerate case of a convex polygon.

Let $P = x_1, x_2, ..., x_n$ be n points on a line with $x_i \le x_{i+1}$, $1 \le i \le n-1$. We say two line segments $x_i x_j$,$x_l x_k$ cross each other if their intervals partially overlap. A minimum weight triangulation of P is a maximal set of non-crossing line segments of P with minimum possible total weight, where the weight of a diagonal $x_i x_j$ is $| x_i - x_j |$. The minimum weight triangulation in this case is just the optimal alphabetic tree with the $n-1$ nodes as the line segments (x_i, x_{i+1}), for $i = 1, 2, ..., n-1$. So we have the following theorem.

Theorem 2.2 *A minimum weight triangulation of n points on a line can be found in $O(n \log n)$ time and $O(n)$ space.*

2.2 MWT for General Point Set

Gilbert and Klincsek's algorithm does not make good use the geometric property of the problem. Lingas, Heath and Pemmaraju extended the algorithm to compute a minimum weight triangulation of a *cell* [38, 27]. A *cell* is any interior face of a straight line planar embedding of a graph. With the extension algorithm, a *dynamic programming* algorithm can be given to compute a minimum weight triangulation for any planar point set with a *cell* constraint.

Anagnostou and Corneil consider how to use the structure of the convex layers of a point set and give a *dynamic programming* to compute a minimum weight triangulation [5], their main result is the following;

Theorem 2.3 *A minimum weight triangulation of n points in the plane can be found in $O(n^{3k+1})$ time, where the n points is restricted on k nested convex polygons.*

Meijer and Rappaport later improved the bound to $O(n^k)$ when S is restricted to lie on k non-intersecting line segments[41].

Xu [47] and Cheng et al.[12] consider the case, when some subgraphs of a minimum weight triangulation is known , and how to connect the disconnected components to get only one connected subgraph of the minimum weight triangulation and use *dynamic programming* for a *cell* to compute the optimal one.

Theorem 2.4 *If a subgraph with k connected components of a minimum weight triangulation of n points in the plane is given, then the complete minimum weight triangulation of the point set can be computed in $O(n^{k+2})$ time.*

The above theorem shows that to find more edges in a minimum weight triangulation will make the possible to find a minimum weight triangulation easy.

2.3 Remarks

The problem of finding a fast algorithm to compute a minimum weight triangulation of a convex polygon is still open. Yao[51] presents a technique by which the time complexity of some dynamic programming algorithms is reduced from $O(n^3)$ to $O(n^2)$. The technique requires the monotonicity of certain bivariate functions, but the minimum weight triangulation problem for a convex polygon does not hold the condition. Even no algorithm is found to compute a minimum weight triangulation in $O(n^3)$ time and $o(n^2)$ space.

3 Subgraphs within MWT

3.1 Intersections of Triangulations

In recent years, there are many results related with the subgraph scheme. The convex hull $CH(S)$ of a point set S is an obvious subgraph of any minimum weight triangulation of a point set. Let E denote the set of all the segments with endpoints in S. A line segment pq with $p, q \in S$ is called a *stable line segment* of all triangulations of S, if no line segment in E properly intersects pq. The intersection of all possible triangulations of S then is the set of all stable line segments in S, denoted by $SL(S)$.

As a combinatorial geometrical problem, some properties of stable line segments of a set of planar points have been investigated in [48]. It is shown that the maximum number of stable line segments in S is $2(n-1)$.

A more important property is the relationship between $SL(S)$ and so-called k-optimal triangulations. $T(S)$ is called a *k-optimal triangulation* for $4 \le k < n$, denoted by $LOT_k(S)$, if every k-sided simple polygon drawn from $T(S)$ is optimally triangulated by some edges of $T(S)$.

Let $SL_k(S)$ denote the intersection of all possible $LOT_k(S)$'s (i.e., the set of edges that are in every $LOT_k(S)$). Let $MWT(S)$ denote a minimum weight triangulation of S. Then, we have that

$$SL(S) \subseteq SL_4(S) \subseteq \cdots \subseteq SL_k(S) \subseteq \cdots \subseteq SL_{n-1}(S) \subseteq MWT(S)$$

In some special cases of S, $SL(S)$ forms a connected graph. Thus, an $MWT(S)$ can be constructed in polynomial time using the dynamic programming algorithm proposed in [38, 47].

So far the structure properties of $SL(S)$ have been studied[47, 48], and it is shown that $SL(S)$ can be found in $O(n^2 \log n)$ time and $O(n)$ space by Mirzaian, Wang and Xu [42].

A subgraph $LMT(S)$ of $SL_4(S)$ proposed by Belleville et al.[6], Dickerson and Montague [17] can be found in $O(n^4)$ time and $O(n^3)$ space [17]. The $LMT(S)$ has much more edges than $SL(S)$ sometimes. But it is pointed out that for uniformly distributed points, the expected number of components is $\Theta(n)$; see Bose et al.[10]. An improved algorithm using $O(n^3 \log n)$ time and $O(n^2)$ was given in [11].

It is easy to show that

$$SL(S) \subseteq LMT(S) \subseteq SL_4(S)$$

3.2 Local Conditions

Subgraphs found from the intersection aspect of $SL_k(S)$ view the subgraph from a global way. In the following we can see that some subgraphs of the minimum weight triangulation can also be found from some local conditions.

It is shown in [25] that the shortest edge between two points in S belongs to any MWT. Keil proves that a much larger graph, $\sqrt{2}$-skeleton, is always a subgraph of a MWT [31]. The $\sqrt{2}$-skeleton is the β-skeleton defined by Kirkpatrick and Radke in [32] for $\beta = \sqrt{2}$. Given two points x and y, define xy to be the edge connecting x and y and define $|xy|$ to be the length of xy. For $\beta \ge 1$, the *forbidden neighborhood* of x and y is the union of two disks with radius $\beta|xy|/2$ that pass through both x and y. Given a point set S and $x, y \in S$, xy belongs to the β-skeleton of S if no point in S lies in the interior of the forbidden neighborhood of x and y. Let α_{xy} be the angle that the chord xy subtends at one of the circles. Then $\beta = 1/\sin \alpha_{xy}$.

It seems that the β-skeleton is a subgraph of a MWT for $\beta \ge 1/\sin(\pi/3)$. Cheng and Xu proved that the β-skeleton is a subgraph of a MWT, for $\beta > 1/\sin \kappa \approx 1.17682$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$ [14].

Yang, You and Xu formulated and proved a different property: if the union of the two disks centered at x and y with radius $|xy|$ is empty, then xy is in a MWT [50].

Note that the subgraph generated by the above condition and the β-skeleton do not contain each other for $\beta > 1/\sin(\pi/3)$, but for $\beta \le 1/\sin(\pi/3)$, the β-skeleton contains the subgraph generated by the above condition.

The proof of the β-skeleton to be a subgraph of a minimum weight triangulation are based on proving an improved version of the key lemma, *Remote Length Lemma*, first given by Keil [31].

Lemma 3.1 *Suppose that $\beta \ge \sqrt{2}$. Let x and y be the endpoints of an edge in the β-skeleton of a set S of points in the plane. Let p, q, r, and s be four other distinct points of S such that pq intersects the interior of xy, rs intersects the interior of xy, pq and rs does not intersect the interior of each other and p and s lie on the same side of the line through xy. Then either $|qr| < |pq|$ or $|qr| < |rs|$.*

All the subgraphs found above have some symmetric local conditions. A sufficient unsymmetric condition is given in [46], but the condition is of a little complicate.

3.3 Remarks

Even some subgraphs of a minimum weight triangulation are found, either form the global way or the local conditions, but to compute all the known subgraphs still can not guarantee that the output is a connected graph or only has a constant number of connected components.

Even some subgraphs of $SL_4(S)$ can be computed efficiently, but it is not known how to compute $SL_4(S)$ in polynomial time.

In [49], it is pointed out that to find more edges in a minimum weight triangulation can improve the performance for some heuristics for the minimum weight triangulation.

4 Matching Properties

4.1 Matching Between Triangulations

For a given point set S in the plane, we know that any triangulation, $T(S)$, of S has the same number of edges, and also the same number of triangles. This fact makes us to find out the matching relationship between different triangulations of S.

With the *Hall condition* of marriage theorem(see [9]), the following theorems was found independent in [2] and [13].

Theorem 4.1 *Let S be a finite set of points in the plane and consider two triangulations R and B of S. There exists a perfect matching between the set of edges of R and the set of edges of B, with the property that matched edges either cross or are identical.*

Theorem 4.2 *Let S be a finite set of points in the plane and consider two triangulations R and B of S. There exists a perfect matching between the set of triangles of R and the set of triangles of B, with the property that matched triangles either overlap or are identical.*

We can impose a stronger condition which requires the matched triangles to share a vertex and the following theorem holds.

Theorem 4.3 *Let S be a finite set of points in the plane and consider two triangulations R and B of S. There exists a perfect matching between the set of triangles of R and the set of triangles of B, with the property that matched triangles*
(a) have common interior points, and
(b) share at least one vertex.

With the above matching theorems, we can obtain some results related with minimum weight triangulations. For a given triangulation $T(S)$ of S, how can we know whether is a minimum weight triangulation?

In [1], it exhibit a class of planar point sets where the minimum-weight triangulation can be computed in polynomial time easy to recognized.

Let us call an edge $e \in E$ *light* if any edge in E that crosses e is longer than e, where E denotes the set of all possible edges with endpoints in S. Light edges obviously do not cross, so the set L of light edges can form at most a triangulation of S.

If L actually is a triangulation then we call L the *light triangulation* of S. Light edges are related to the *greedy triangulation*, which is obtained by iteratively inserting the shortest edge of E that does not cross previously inserted edges. All light edges are contained in the greedy triangulation: a light edge e can never be blocked by previously inserted edges as E does not contain any shorter edge crossing e. Thus, if a light triangulation exists, it is identical to the greedy triangulation, and it is easy to prove length optimality with the above matching theorems.

And we can easily prove the following results; The first one gives a lower bound for the minimum weight triangulation, the second one gives us a way to indentify a triangulation is a minimum weight triangulation for some special point sets.

Theorem 4.4 $w(L) \leq w(MWT(S))$
where $w(L)$ denotes the total weight of the light edges and $w(MWT(S))$ denotes the weight of the minimum weight triangulation.

Theorem 4.5 *If a planar point set S admits a light triangulation L then L is the minimum weight triangulation for S.*

4.2 Matching Among a Triangulation

To comsider the relationship between the point set S and the edge set of a triangulation $T(S)$, with the *Hall condition*'s sufficient and necessary condition for a matching(see [9]), the following theorem was found in [4].

Theorem 4.6 *Let $T(S)$ be an arbitrary triangulation of S, and S^3 denote set of triple copy of S (i.e., to take $p \in S$ three times). Then there is matching from edges of $T(S)$ to S^3 such that if $p \in S^3$ is matched with $e \in T(S)$, then p is an endpoint of e.*

Another version of the above theorem is

Theorem 4.7 *Let $T(S)$ be an arbitrary triangulation of S. Then the edges of $T(S)$ can be oriented such that each point $p \in S$ has an in-degree of at most 3.*

With the above matching property, an approximating property of the well-known greedy triangulation $GT(S)$ of a finite point set S is obtained. Exploiting the concept of so-called light edges, we introduce a definition of $GT(S)$ that does not rely on the length ordering of the edges. Rather, it provides a decomposition of $GT(S)$ into levels, and the number of levels allows us to bound the total edge length of $GT(S)$. In particular, it is shown in [4] that $|GT(S)| \leq 3 \cdot 2^{k+1}|MWT(S)|$, where k is the number of levels and $MWT(S)$ is the minimum weight triangulation of S. Various algorithms for computing the $GT(S)$ are known, and the $GT(S)$ has been used in several applications. See, e.g., [16] for a short history.

One use of the greedy triangulation is a length approximation to the minimum weight triangulation. For a given point set S. Although the $GT(S)$ tends to be short in practical applications, and is provably short for uniformly distributed point sets [36] and for point sets in convex position [37], its worst-case length behaviour is fairly bad. The $GT(S)$ can be a factor of $\Omega(\sqrt{n})$ longer than the MWT; see [34]. Only very recently, a matching upper bound has been proved [35].

In particular, an edge which is not crosse by any shorter edge will surely belong to $GT(S)$. Let us call an edge *light* in this case. Below is a catalog of basic properties of light edges.

Lemma 4.8 *Let L denote the set of all light edges defined by S.*
(a) L is a non-crossing set of edges.
(b) L contains all edges bounding the convex hull of S.
(c) L is a subset of $GT(S)$.
(d) In general, L is not a subset of $MWT(S)$.

In conjunction with the above Lemma, Lemma immediately implies: if L happens to form a triangulation of S, then $|L| = |GT(S)| = |MWT(S)|$.

In any case, we learn that at least a subset of the edges in $GT(S)$ can be bounded in length by the weight of $MWT(S)$.

The edges in L are called *light of level 1*. Let E be the total set of edges defined by S, and let C_1 collect all edges of E that are crossed by some edge in L. Notice that each edge in L, and therefore no edge in C_1, appears in $GT(S)$. Define $E_2 = E \setminus (L \cup C_1)$. An edge $e \in E_2$ is called *light of level 2* if e is not crossed by a shorter edge in E_2. Let L_2 be the set of all edges which are light of level 2, and let C_2 collect all edges of E_2 that are crossed by some edge in L_2. Again, each edge in L_2, and therefore no edge in C_2, appears in $GT(S)$. By setting $E_3 = E_2 \setminus (L_2 \cup C_2)$ we now can define, in the obvious way, the set L_3 of edges which are light of level 3. Repeating this process until $E_{k+1} = \emptyset$ yields a hierarchy of levels $L_1, L_2, \ldots, L_k$ with $L_1 = L$.

It is evident that levels are pairwise disjoint, and that no edge of level i can cross an edge of level j, for $1 \leq i, j \leq k$. More specifically, we have:

Lemma 4.9 $GT(S) = L_1 \cup L_2 \cup \ldots \cup L_k$.

With the above Lemma and the matching theorem in this section, the following theorem is proved in [4].

Theorem 4.10 *Let S be a finite set of points in the plane, and let k be the number of levels attained by $GT(S)$. Then $|GT(S)| \leq c_k \cdot |MWT(S)|$, where $c_1 = 1$ and $c_k = 3 \cdot 2^{k+1}$ for $k \geq 2$.*

4.3 Remarks

The matching theorems between triangulations may be generalized to the framework of independence systems. An *independence system* $\mathcal{I}$ is a non-empty collection of subsets of a ground set E which is closed under taking subsets: if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$. The elements of $\mathcal{I}$ are called the *independent* sets, the remaining subsets of E are called *dependent.* A *circuit* of $\mathcal{I}$ is a minimal dependent set.

In the triangulation case, a set of non-crossing edges (or of non-overlapping triangles) may be considered independent. The circuits of this independence system have two elements; they are the pairs of crossing edges (or of overlapping triangles respectively).

Theorem 4.11 *Let $R \in \mathcal{I}$ be any independent set, and let $B \in \mathcal{I}$ be an independent set of maximum cardinality in $\mathcal{I}$. Then there is an injective* *mapping* $g: R \to B$ *such that for every element* $e \in R$ *we have* $g(e) = e$, *or* $\{g(e), e\}$ *is contained in a circuit.*

The matching theorem for the point set and the edge set of a triangulation can also be extented to a more general case. For a graph $G = (V, E)$, if we define its local density as $d = \max \frac{|E'|}{|V'|}$, for any subgraph $G' = (V', E')$ of G. Then the following theorem holds [3];

Theorem 4.12 *Let* $G = (V, E)$ *be an simple graph with local density* d, *then the edges of* G *can be oriented such that each vertex* $v \in V$ *has an in-degree of at most* d.

5 Heuristics

Since the complexity statue of finding a minimum weight triangulation is unsolved, some heuristics and approximate algorithms were considered. Shamos and Hoey[45] present a divide and conquer algorithm to construct a *Voronoi diagram* of n points in a plane in $O(n \log n)$ time. This implies that the *Delaunay triangulation*, which is the planar dual of the *Voronoi diagram*, can be constructed in $O(n \log n)$ time [45]. A *greedy triangulation* of a point set S is obtained by inserting compatible edges in increasing length order, where an edge is compatible if it does not cross previously inserted ones and it can be found in $O(n^2 \log n)$ time and $O(n)$ space [26]. Shamos and Hoey state that both the *greedy* and the *Delaunay triangulations* are *minimum weight triangulations* , and hence the *Delaunay triangulation* can be computed more efficient than the *greedy triangulation*[45] . Lloyd provides counterexamples to show that both the *Delaunay triangulation* and the *greedy triangulation* are not always the *minimum weight triangulations* [40]. In fact, his counterexamples show that neither triangulation is opimal even for a convex polygon.

Some heuristics are also investigated. Lingas gave a heuristic using the two steps strategy that first find a *minimum spaning tree*, MST, of S and then optimally triangulate the cell with the MST constraint [38]. Heath and Pemmaraju give a similar heuristic which first find $GT(S)$ of S and the optimal triangulate the cell with the MST of $GT(S)$ [27]. Both heuristics run in $O(n^3)$ time and have some good experimental statue.

Plaisted and Hong presents an algorithm and show that the weight of the triangulation which their algorithm produces is within $O(\log n)$ of the weight of the optimal triangulation [43].

None of the above heuristics can guarantee to produce a triangulation within a constant ratio of the minimum weight triangulation in the worst case. Levcopoulos and Krznaric gave a modified greedy triangulating algorithm and proved that the output of their algoithm has a constant ratio with the optimal ones[35].

6 Conclusion

There are still some open problems related with the minimum weight triangulation problem.

The most important one is to determine the complexity statue of finding a minimum weight triangulation.

The problem of finding an algorithm to compute a minimum weight triangulation of a convex polygon in less than $O(n^3)$ is of more interesting and has more applications.

As we know that to find a local optimal triangulation of a point set is relative easy, since the greedy triangulation is a local optimal one. Up to now, there is no any result concern with how to compute a 5-optimal triangulation in polynomial time.

To find triangulations with some another optimality criteria are still an interesting topic. There is no any results on the problem to find a triangulation with minmax area and the problem to find a triangulation with minmax perimeter.

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